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Raman Laser Modeling: Mathematical and Numerical Analysis

François Castella* , Philippe Chartier† , Erwan Faou ‡
Dominique Bayart§ , Florence Leplingard¶ , Catherine Martinelli‖

Thème 4 — Simulation et optimisation
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Abstract: In this paper we study a discrete Raman laser amplification model given as a Lotka-Volterra system. We show that in an ideal situation, the equations can be written as a Poisson system with boundary conditions using a global change of coordinates. We address the questions of existence and uniqueness of a solution. We deduce numerical schemes for the approximation of the solution that have more stability than the standard shooting methods.

Key-words: Lotka-Volterra, Poisson system, Hamiltonian system, Boundary value problem

* IRMAR, Université de Rennes 1, Campus Beaulieu 35042 Rennes Cedex, France (`Francois.Castella@univ-rennes1.fr`)
† INRIA Rennes, Campus Beaulieu 35042 Rennes Cedex, France (`Philippe.Chartier@irisa.fr`)
‡ INRIA Rennes, Campus Beaulieu 35042 Rennes Cedex, France (`Erwan.Faou@irisa.fr`)
§ ALCATEL Research & Innovation, Unité Transmissions Photoniques, Route de Nozay, 91460 Marcoussis, France (`Dominique.Bayart@alcatel.fr`)
¶ ALCATEL Research & Innovation, Unité Transmissions Photoniques, Route de Nozay, 91460 Marcoussis, France (`Florence.Leplingard@alcatel.fr`)
‖ ALCATEL Research & Innovation, Unité Transmissions Photoniques, Route de Nozay, 91460 Marcoussis, France (`Catherine.Martinelli@alcatel.fr`)

Unité de recherche INRIA Rennes
IRISA, Campus universitaire de Beaulieu, 35042 Rennes Cedex (France)
Téléphone : +33 2 99 84 71 00 —Télécopie : +33 2 99 84 71 71

Modélisation de laser Raman : analyse mathématique et numérique

Résumé : Nous étudions un modèle discret d'amplification dans un laser Raman, écrit comme un système de Lotka-Volterra. Nous montrons que dans une situation idéale, les équations peuvent s'écrire comme un système de Poisson avec des conditions au bord en utilisant un changement de coordonnées global. Nous étudions les questions d'existence et d'unicité de la solution. Nous en déduisons des algorithmes pour l'approximation numérique de la solution qui sont plus stables que les méthodes de tir classiques.

Mots-clés : Lotka-Volterra, Système de Poisson, Système Hamiltonien, Problème avec conditions aux bord

1 Introduction

The problem described in this paper originates from a model of Raman laser amplification effect in an optic fiber [6]. Standard discrete models of this phenomenon (see [1] or [7]) lead to a system of differential equations of Lotka-Volterra form (see for instance [5]), where high-frequency waves traveling forward and backward in the fiber disseminate part of their energy to low-frequency waves through a prey-predator process. Boundary conditions corresponding to Bragg reflecting lattices are imposed on both sides of the laser cavity [7].

In the case of an idealized fiber, this system turns out to have a Poisson structure (see for instance [5]) for which we can exhibit explicitely the Hamiltonian and the Casimir invariants. However, the underlying Hamiltonian function is affine with respect to the unknowns. The corresponding invariant manifold is thus not compact so that the existence of a solution remains a non-trivial question. Moreover, the system is posed as a boundary value problem. These aspects contribute to make a numerical approximation difficult to obtain : for instance, the *shooting* method [2] is to be banned here due to the presence of nonlinearities (most initial values would lead to blow-up in finite time); more elaborated methods, such as finite differences, collocation, or multiple shooting, are possible alternatives, but might become prohibitively costly in large dimension.

Another difficulty comes from the fact that in the original variables, there exists always a "trivial" solution corresponding to the case where the Raman amplification effect has not yet started. Numerically, the presence of this dummy solution makes the choice of the initial values in an iterative process difficult to determine.

Here we prove that the Poisson system can be brought to canonical form through a *global* change of coordinates. Note that the change of coordinates defined in Darboux-Lie's Theorem is usually local and that the literature offers only a few examples of such global changes (see [5] pp. 241 for a nice example). We show that for an ideal fiber the equations can be written

$$u' = G\nabla_u H(u,d) \quad \text{with} \quad H(u,d) = \sum_{i=1}^{n} d_i \sinh u_i, \tag{1.1}$$

where u is an unknown vector of dimension $n \geq 1$ of functions defined on the fiber, d an unknown element of $\mathbb{R}^n$, G a skew-symmetric matrix and $H(u,d)$ the hamiltonian of the problem. At this stage, getting a canonical Poisson system requires only to bring the *constant skew-symmetric* matrix G to canonical form. Note that the d_i's are Casimir invariants of the underlying Poisson structure (see [5]).

In this form, the "trivial" solution has disappeared, but the problems of existence and uniqueness of the solution (and thus definition and convergence of shooting schemes) are still present. Note that the boundary conditions depend also on the unknown values of the Casimir invariants d_i. In the general case (i.e. not for an idealized fiber), we show that we can write the problem in a form close to (1.1) where the d_i's remain invariants of the problem with unknown values.

We show that it is actually possible to take benefit of the available free parameters d so as to *reformulate* the problem as a *Cauchy problem for a system of integro-differential equations*. In this form, the problem is well-posed : using standard techniques (Schauder's theorem), the existence of solutions can be easily proved for boundary conditions independent of d (see [3]). Uniqueness for boundary values that are not too far apart and an arbitrary dimension is also shown. Note that ad-hoc techniques allow for the treatment of the one and two-dimensional cases for arbitrary boundary values (see [3]). Eventually, we prove the existence and uniqueness of a solution to the original problem (with boundary conditions depending on d) under strong assumptions on the data.

Using the integro-differential formulation of the problem, we derive a numerical Picard-like scheme converging toward the solution under smallness assumptions on the data. We conclude this work by giving numerical examples showing that this scheme converges linearly to the solution in practical cases.

In Section 2, we describe the original Lotka-Volterra equations and in Section 3 we exhibit the Poisson structure in the case of an ideal fiber. We then show a global version of the Darboux-Lie Theorem for this system in Section 4.

Sections 5, 6, 7 are devoted to the proof of existence and uniqueness results for the general problem using the change of unknowns defined in Section 4. In practical cases, the matrix G is invertible when n is even, and singular with the eigenvalue 0 of multiplicity 1 when n is odd. We thus distinguish these two cases. In Section 5, we first consider the simplest case where n is even, G is invertible and the boundary conditions are independent of d. We combine this result with the use a fixed-point theorem to obtain an existence and uniqueness result for the general case (i.e. with boundary conditions depending on d) when n is even in Section 6. Eventually, Section 7 deals with the case where n is odd.

Finally we give numerical results in Section 8.

2 A model of cascaded Raman fiber laser

We denote by L the length of the cavity, and we suppose that n rays at given frequencies ν_1, ν_2, ..., ν_n are represented by n functions $F_i(x)$ and $B_i(x)$ for $x \in [0, L]$ denoting the powers of the *forward* and *backward* waves respectively.

The model equations can be written as follows, where the index i runs from 0 to n (see [7] and [1]):

$$\begin{aligned} F_i' &= -\alpha_i F_i - \textstyle\sum_{j<i} g_{ij}(F_j + B_j)F_i + \sum_{j>i} g_{ij}(F_j + B_j)F_i, \\ B_i' &= \alpha_i B_i + \textstyle\sum_{j<i} g_{ij}(F_j + B_j)B_i - \sum_{j>i} g_{ij}(F_j + B_j)B_i. \end{aligned}$$

Here and in the sequel, the $'$ denotes the derivation with respect to $x \in [0, L]$. The coefficients g_{ij} are non negative and represent the Raman gain between the wave length of the level i and j. The coefficients $\alpha_i > 0$ are attenuation coefficients. We define the Raman gain matrix $G = (G_{ij})$ by:

$$\begin{array}{rcll} G_{ij} &=& -g_{ij} & \text{if} \quad j > i, \\ G_{ij} &=& g_{ij} & \text{if} \quad i < j, \\ G_{ii} &=& 0. & \end{array}$$

We can now rewrite equations (2.1) in a more compact Lotka-Volterra form as follows:

$$\begin{aligned} F_i' &= -\alpha_i F_i + \textstyle\sum_{j=1}^n G_{ij}(F_j + B_j)F_i \\ B_i' &= \alpha_i B_i - \textstyle\sum_{j=1}^n G_{ij}(F_j + B_j)B_i. \end{aligned} \tag{2.1}$$

To complete the description of the problem, it remains to consider the boundary conditions in 0 and L. They read

$$F_1(0) = P \quad \text{and} \quad F_i(0) = R_i^0 B_i(0), \quad i = 2, \ldots, n \tag{2.2}$$

and

$$B_i(L) = R_i^L F_i(L), \quad i = 1, \ldots, n-1 \quad \text{and} \quad F_n(L) = R_{\text{out}} B_n(L), \tag{2.3}$$

where the coefficients R_i^0 and R_i^L are reflectivity coefficients of the Bragg lattices in $x = 0$ and $x = L$ respectively, and R_{out} is the last reflectivity coefficient (see [7]). The number P is given and represent the pump power injected in the cavity at the frequency ν_1. We will mainly consider the situation where $R_i \simeq 1$ and $R_{\text{out}} < 1$) (usually $R_{\text{out}} = 0.15$ and $R_i^0 = R_i^L = 0.99$).

Note that the system (2.1-2.2-2.3) possesses the "trivial" solution

$$F_1(x) = P\exp(-\alpha_1 x), \qquad B_1(x) = R_1^L P\exp(\alpha_1(x-2L))$$

and $F_i = B_i = 0$ for $i \geq 2$. This solution corresponds to the case where the Raman amplification effect has not yet appeared. Indeed, the system (2.1) describes a stationary regime of more general time dependent equations. In practice, the laser starts on the noise due to a further term not present in equations (2.1-2.2-2.3), the so-called "Amplified Spontaneous Emission" (ASE) (see [6]). From a mathematical point of view, when the (ASE) term is taken into account, the only admissible stationary regime is the non-trivial one. However, as soon as the laser starts, the contribution of the (ASE) can be completely neglected. We are thus looking for a physical solution of (2.1-2.2-2.3), satisfying the further assumptions:

$$F_i > 0 \quad \text{and} \quad B_i > 0 \quad \text{for} \quad i = 1, \dots, n. \tag{2.4}$$

Even at this early stage, it is interesting to notice that the system has several mathematical invariants. A simple calculation shows indeed that

$$\forall\, i = 1, \dots, n, \quad \forall\, x \in [0, L], \quad (F_i B_i)(x) = (F_i B_i)(0) = (F_i B_i)(L).$$

If we make the further assumption that G is skew-symmetric (that is to say that there is no loss of gain), and that the α_i are all vanishing (meaning that there is no loss of energy due to the Joule effect within the fiber), then we can further notice that $\sum_j (F_j - B_j)$ is kept constant along the fiber. This quantity can be interpreted as the energy of the system and its preservation in absence of attenuation is physically sounded.

Remark 2.1 In practical cases, the matrix G is close to a bidiagonal matrix $\tilde{G}$ such that $\tilde{G}_{ij} = 0$ for $|i-j| > 1$, $\tilde{G}_{ii} = 0$ for $i = 1, \dots, n$, $\tilde{G}_{i,i+1} = -\sigma$ for $i = 1, \dots, n-1$ and $\tilde{G}_{i+1,i} = \sigma$ for $i = 2, \dots, n$, where σ is a real positive number. Note that with this definition, $\tilde{G}$ is invertible when n is even and singular with the eigenvalue 0 of multiplicity 1 when n is odd. This corresponds to the case where we only take into account the interactions between successive frequencies, and where we suppose that the value of the Raman gain does not depend on the frequencies, but only on the difference between two frequencies (see [7]). In this special cases, we will see that we can obtain more refined estimates. ■

The existence of these invariants will become natural in the next section, where the system will be shown to have a Poisson structure. It is a well-known fact that such systems can be brought back to canonical form, through a local change of variables. In the context of the present study, it is in fact possible to exhibit a *global* change of variables and this will be the subject of Section 4.

3 A conservative model with Poisson structure

In this section, we consider a somewhat idealized model, which can be viewed as a simplified form of the previous one. The so-obtained system has obviously the advantage to be more tractable from a mathematical point of view. From now on, we thus make the following assumptions :

1. $g_{ij} = -g_{ji}$, so that the matrix G is skew-symmetric.
2. the matrix G is of maximal rank : G is invertible if n is even and G is of rank $n-1$ is n is odd.
3. $\alpha_i = 0$ for all $i = 1, \dots, n$ (there is no loss of energy within the fiber).

In the following we set $Y = (F, B) \in \mathbb{R}^{2n}$ and we define $G(F, B)$ as being the $n \times n$ matrix with coefficients $G_{ij} F_i B_j$. The $2n$-dimensional square-matrix $J(Y)$ is then constructed by the equation

$$J(Y) := \begin{pmatrix} G(F,F) & -G(F,B) \\ -G(B,F) & G(B,B) \end{pmatrix}, \tag{3.1}$$

and is clearly skew-symmetric. We write $J_{\alpha\beta}(Y)$ the coefficients of this matrix ($\alpha, \beta = 1, \dots, 2n$). Now for two functions H and K of Y we define the bracket

$$\{H, K\}(Y) = \sum_{\alpha,\beta=1}^{2n} \frac{\partial H(Y)}{\partial Y_\alpha} J_{\alpha\beta}(Y) \frac{\partial K(Y)}{\partial Y_\beta}. \tag{3.2}$$

We will see that this bracket defines a *Poisson bracket*, that is satisfies the three identities, for all H, K and Q functions of Y:

$$\begin{aligned} &\{H, K\} = -\{K, H\} && \text{(skew-symmetry)} \\ &\big\{\{H, K\}, Q\big\} + \big\{\{K, Q\}, H\big\} + \big\{\{Q, H\}, K\big\} = 0 && \text{(Jacobi identity)} \\ &\{H \cdot K, Q\} = H \cdot \{K, Q\} + K \cdot \{H, Q\} && \text{(Leibniz rule)} \end{aligned}$$

This is a consequence of the following lemma (with $m = 2n$):

Lemma 3.1 *Let $m \geq 1$ and $A_{\alpha\beta}$ be a skew-symmetric matrix of order m ($A_{\alpha\beta} = -A_{\beta\alpha}$). Let $A(Y)$ be the matrix of order m with coefficients $A_{\alpha\beta} Y_\alpha Y_\beta$. Then the application*

$$(H, K) \mapsto \sum_{\alpha,\beta=1}^{m} \frac{\partial H(Y)}{\partial Y_\alpha} A_{\alpha\beta}(Y) \frac{\partial K(Y)}{\partial Y_\beta}.$$

defines a Poisson Bracket.

PROOF. It is well known (see [5]) that the result is true if the matrix $A(Y)$ satisfies the relation

$$\sum_{\alpha=1}^{m} \Big(\frac{\partial A_{\beta\sigma}(Y)}{\partial Y_\alpha} A_{\alpha\delta}(Y) + \frac{\partial A_{\sigma\delta}(Y)}{\partial Y_\alpha} A_{\alpha\beta}(Y) + \frac{\partial A_{\delta\beta}(Y)}{\partial Y_\alpha} A_{\alpha\sigma}(Y) \Big) = 0, \tag{3.3}$$

for all $Y \in \mathbb{R}^m$. But we have

$$\sum_{\alpha=1}^{m} \frac{\partial A_{\beta\sigma}(Y)}{\partial Y_\alpha} A_{\alpha\delta}(Y) = A_{\beta\sigma} A_{\beta\delta} Y_\beta Y_\delta Y_\sigma + A_{\beta\sigma} A_{\sigma\delta} Y_\beta Y_\delta Y_\sigma$$

Thus the relation (3.3) is equivalent to

$$A_{\beta\sigma} A_{\beta\delta} + A_{\beta\sigma} A_{\sigma\delta} + A_{\sigma\delta} A_{\sigma\beta} + A_{\sigma\delta} A_{\delta\beta} + A_{\delta\beta} A_{\delta\sigma} + A_{\delta\beta} A_{\beta\sigma} = 0$$

and we see that this relation is satisfied using the fact that $A_{\alpha\beta}$ is skew-symmetric. ■

Now we see that $J(Y)$ is of this form, using the partition $Y = (F, B)$ and thus (3.2) defines a Poisson Bracket. A simple computation then yields the following result:

Proposition 3.2 *Suppose that all the $\alpha_i = 0$ and that G is skew-symmetric. Then the system* (2.1) *is equivalent to the system*

$$Y' = J(Y) \nabla H_0(Y) \tag{3.4}$$

where $J(Y)$ is the matrix (3.1) *and $H_0(Y)$ is the hamiltonian*

$$H_0(Y) = \sum_{i=1}^{n} (F_i - B_i) \quad \textit{for} \quad Y = (F, B). \tag{3.5}$$

Remark that if n is even and if the α_i are non zero, then there exist n real coefficients a_i such that the system (2.1) is equivalent to the system

$$Y' = J(Y)\nabla H_a(Y) \tag{3.6}$$

where $J(Y)$ is the matrix (3.1) and $H_a(Y)$ is the hamiltonian

$$H_a(Y) = \sum_{i=1}^{n} \big(F_i - B_i + a_i \log F_i\big) \quad \text{for} \quad Y = (F, B). \tag{3.7}$$

Indeed, we see that we have

$$\nabla H_a(Y) = (1 + a_1/F_1, \ldots, 1 + a_n/F_n, -1, \ldots, -1)^T.$$

Thus the system (2.1) is equivalent to the system (3.6) if and only if we have $a^T G = -\alpha^T$ where α denotes the vector $(\alpha_i)_{i=1,\ldots,n}$ and a the vector $(a_i)_{i=1,\ldots,n}$. Using the assumption that G is invertible when n is even, we get the result.

The Poisson structure of the problem yields natural invariants of the problem.

Proposition 3.3 *Suppose that $\alpha_i = 0$ and G is skew-symmetric of maximal rank. The system* (2.1) *is thus equivalent to the Poisson system* (3.4).

If n is even the system possesses n Casimir invariants $c_i = F_i B_i$, $i = 1, \ldots, n$.

If n is odd the system possesses $n+1$ Casimir invariants $c_i = F_i B_i$, $i = 1, \ldots, n$, and $\sum_{i=1}^n a_i \log F_i$ where $a = (a_i)_{i=1}^n$ is such that $a^T G = 0$.

Proof. By construction, if $Y_\alpha \neq 0$ for $\alpha = 1, \ldots, n$, the matrix $J(Y)$ defined in (3.1) has the same rank as the matrix G. Moreover, if we set $c_i = F_i B_i$ for $i = 1, \ldots, n$, we see that the vectors ∇c_i are in the kernel of $J(Y)$. As these vectors are linearly independent, the c_i, $i = 1, \ldots, n$, are n Casimirs of the system.

If n is even, there is no other Casimir because the rank of $J(Y)$ is n for generic non zero Y_α, $\alpha = 1, \ldots, n$.

If n is odd and if we set $A(Y) = \sum_{i=1}^n a_i \log F_i$, then we have

$$\nabla A(Y) = (a_1/F_1, \ldots, a_n/F_n, 0, \ldots, 0)^T,$$

and we compute directly that

$$\big(\nabla A(Y)^T J(Y)\big)_\alpha = \begin{cases} (a^T G)_\alpha F_\alpha & \text{if} \quad 1 \leq \alpha \leq n, \\ (a^T G)_{\alpha - n} B_\alpha & \text{if} \quad n+1 \leq \alpha \leq 2n, \end{cases}$$

and hence $\nabla A(Y)^T J(Y) = 0$. As $\nabla A(Y)$ is independent of the ∇c_i for generic Y, this means that $A(Y)$ is the last Casimir of the system. ■

Suppose now that $\alpha_i \neq 0$. For given i we compute that

$$\begin{aligned} F_i' B_i &= -\alpha_i F_i B_i + \textstyle\sum_{j=1}^n G_{ij}(F_j + B_j) F_i B_i \quad \text{and} \\ B_i' F_i &= \alpha_i B_i F_i - \textstyle\sum_{j=1}^n G_{ij}(F_j + B_j) B_i F_i. \end{aligned}$$

and thus it is clear that $F_i' B_i + F_i B_i' = 0$, and hence the $c_i := F_i B_i$ are constant along the trajectories of (2.1). Thus the system the system (2.1) with $\alpha_i \neq 0$ still possesses the n invariants $c_i = F_i B_i$ for $i = 1, \ldots, n$.

4 Change of unknowns

The system (2.1) can be seen as the expression of the derivative of $\log(F_i)$ for $i = 1, \dots, n$. Moreover, we know that in all cases, the products $c_i := F_i B_i$ are invariants for $i = 1, \dots, n$. That is why we make the following change of unknowns:

$$(F, B) \mapsto \left\{ \begin{array}{lcl} c_i & = & F_i B_i, \\ u_i & = & \log(F_i/\sqrt{c_i}) \end{array} \right. \qquad i = 1, \dots, n. \tag{4.1}$$

This change of unknowns is well defined under the assumption (2.4) and we have the inverse relation

$$F_i = \sqrt{c_i} e^{u_i} \quad \text{and} \quad B_i = \sqrt{c_i} e^{-u_i}.$$

It is thus clear that we have $F_i + B_i = 2\sqrt{c_i}\cosh(u_i)$ and $u_i' = F_i'/F_i$. The system (2.1) can thus be written,

$$\left\{ \begin{array}{lcl} u_i' & = & -\alpha_i + 2\sum_{j=1}^n G_{ij}\sqrt{c_j}\cosh u_j \\ c_i' & = & 0 \end{array} \right. \qquad \text{for} \quad i = 1, \dots, n. \tag{4.2}$$

The boundary conditions are now

$$u_1(0) = \log(P/\sqrt{c_1}) \quad \text{and} \quad u_i(0) = \tfrac{1}{2}\log R_i^0 \quad \text{for} \quad i = 2, \dots, n, \tag{4.3}$$

and

$$u_i(L) = -\tfrac{1}{2}\log R_i^L \quad \text{for} \quad i = 1, \dots, 0 \quad \text{and} \quad u_n(L) = -\tfrac{1}{2}\log R_{\text{out}}. \tag{4.4}$$

Let us consider now the ideal case, where $\alpha_i = 0$ and G is skew-symmetric of maximal rank. In the new coordinates, the Hamiltonian $H_0(Y)$ in (3.5) writes now

$$H_0(u, c) = 2\sum_{i=1}^n \sqrt{c_i}\sinh u_i.$$

The classical Darboux-Lie theorem states that a Poisson system $y' = A(y)\nabla H(y)$ can be locally written as a system of the form

$$z' = J_0 \nabla K(z) \quad \text{with} \quad J_0 = \begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.5}$$

where J is the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the identity matrix of dimension d, where $2d$ is the rank of $A(y)$.

Here, for $F_i > 0$ and $B_i > 0$, $i = 1, \dots, n$, the matrix $J(Y)$ is of rank n if n is even and $n - 1$ if n is odd. We now show that using the previous change of unknown, we can write a *global* Darboux-Lie transformation. We first have the lemma:

Lemma 4.1 *There exist a invertible matrix M of order n such that*

$$G = MAM^T$$

where $A = J^{-1}$ if n is even, and

$$A = \begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

where J^{-1} is of dimension $n - 1$ if n is odd.

PROOF. By the real Schur decomposition theorem (see for instance [4]), any real matrix G can be brought to the form

$$Q^T G Q = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ 0 & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & R_{mm} \end{bmatrix}$$

where the R_{ii} are either a 1-by-1 real matrix or a 2-by-2 real matrix with complex conjugate eigenvalues. Note that the matrix Q involved in the transformation is real orthogonal, i.e. satisfies $Q^T Q = I$. Now, since G is supposed to be skew-symmetric, we immediately get

$$\begin{aligned} R_{ij} &= 0 \quad \text{for } j \neq i, \\ R_{ii}^T &= -R_{ii}. \end{aligned}$$

This means that $Q^T G Q$ is in fact a block-diagonal matrix with skew-symmetric blocks of dimension 1 or 2 on the diagonal, i.e. of the form $R_{ii} = 0$ or

$$R_{ii} = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}.$$

Consider now the block diagonal matrix D with diagonal blocks $D_{ii} = 1$ if $R_{ii} = 0$ and

$$D_{ii} = \begin{bmatrix} |\omega_i|^{-1/2} & 0 \\ 0 & |\omega_i|^{-1/2} \end{bmatrix},$$

otherwise. The rescaled matrix $D^T Q^T G Q D$ is block-diagonal with diagonal blocks $D_{ii} R_{ii} D_{ii}$ either null or of the form

$$\begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix}.$$

It remains to notice that $D^T Q^T G Q D$ can be brought to the form stated in the lemma through a permutation matrix P satisfying $P^T P = I$ (M is then $(QDP)^{-T}$). ■

Using this lemma, we obtain easily the following result:

Theorem 4.2 *Suppose that $\alpha_i = 0$ and G is skew symmetric of maximal rank. Let M be the matrix of lemma 4.1, and let c and u the unknowns defined in (4.1). Then the change of variable $(F, B) \mapsto (v, c)$ where $v = M^{-1} u$ is a global Darboux-Lie change of variable from*

$$\{\, (F, B) \in \mathbb{R}^{2n} \,|\, F_i > 0 \quad \textit{and} \quad B_i > 0 \,\} \quad \textit{to} \quad \{\, (v, c) \in \mathbb{R}^{2n} \,|\, c_i > 0 \,\}.$$

In the coordinate system $z = (v, c)$, the system writes

$$z' = J_0 \nabla K(z),$$

where J_0 is of the form (4.5) with J of dimension n if n is even and $n-1$ if n is odd. The hamiltonian $K(z)$ writes:

$$K(z) = K(v, c) = \sum_{i=1}^{n} \sqrt{c_i} \sinh((Mv)_i) = (\sqrt{c})^T \sinh(Mv).$$

Note that if n is odd, using the definition of M, the last component of v is of the form $\sum_{i=1}^{n} a_i u_i$ where $a = (a_i)_{i=1}^{n}$ is in the kernel of G. Thus v_n is the Casimir of Proposition 3.3.

Going back to the general case (i.e. G non skew-symmetric and $\alpha_i \neq 0$), we see that we reduced the problem (2.1-2.2-2.3) posed on the set $\{ (F, B) \in \mathbb{R}^{2n} \,|\, F_i > 0 \quad \text{and} \quad B_i > 0 \}$ to the problem with boundary conditions (4.2-4.3-4.4) on the set $\{ (u, c) \in \mathbb{R}^{2n} \,|\, c_i > 0 \}$.

We will now study this last problem in several steps. First of all, we note the presence of the inhomogeneous boundary condition $u_1(0) = \log P/\sqrt{c_1}$ depending on c_1 while the others are independent of the coefficients c_i with unknown values. As we will see, this condition is not treated in the same way for n even or n odd.

When n is even we use the following strategy: we study first the problem (4.2-4.3-4.4) but with a Dirichlet condition on $u_1(0)$ independent of c_1 (we write it $\frac{1}{2}\log R_{\text{in}}$ where $R_{\text{in}} > 0$ for homogeneity reasons). This problem is studied in the next section. Once we proved existence and uniqueness result for this problem for "small" values of the boundary conditions, we then give in section 6 an existence and uniqueness result for the initial problem with the boundary condition (4.3). As we will see, the conditions on the power P to obtain this result are severe.

In the case where n is odd, we will see in section 7 that we can use an element of the kernel of G to construct an invariant (the last Casimir invariant in the case where G is skew-symmetric) and that we can obtain directly an expression of c_1 with respect to the data of the problem. Using this, we can prove the existence and uniqueness of the problem (4.2-4.3-4.4) for a given P satisfying strong conditions.

5 Existence and uniqueness results for n even: A modified problem

If n is even, the problem (4.2) together with the boundary conditions (4.3-4.4) seems difficult to solve, because of the presence of the term $\log(P/\sqrt{c_1})$ in (4.3) for which it is hard to have bounds (see the next section). Here, we first prove the existence of a solution of a modified problem, written

$$\left\{ \begin{array}{lcl} u_i' & = & -\alpha_i + 2\sum_{j=1}^{n} G_{ij}\sqrt{c_j}\cosh u_j \\ c_i' & = & 0 \end{array} \right. \quad \text{for} \quad i = 1, \ldots, n, \tag{5.1}$$

with the boundary conditions

$$u_1(0) = \tfrac{1}{2}\log R_{\text{in}} \quad \text{and} \quad u_i(0) = \tfrac{1}{2}\log R_i^0 \quad \text{for} \quad i = 2, \ldots, n, \tag{5.2}$$

and

$$u_i(L) = -\tfrac{1}{2}\log R_i^L \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad u_n(L) = -\tfrac{1}{2}\log R_{\text{out}}, \tag{5.3}$$

where R_{in} is a real number. In the original variable, this corresponds to the boundary condition $F_1(0) = R_{\text{in}} B_1(0)$. In the following, in order to get compact formulas we set:

$$R_1^0 := R_{\text{in}} \quad \text{and} \quad R_n^L := R_{\text{out}}.$$

In the ideal case where G is skew-symmetric and $\alpha = 0$, this problem is a Poisson system with boundary conditions. Using the same method as in [3], we show now that in the general case, we can reformulate this problem as a Cauchy problem for system of integro-differential equations.

Let us integrate the equations (5.1) from 0 to L. We find, for all $i = 1, \ldots, n$:

$$u_i(L) - u_i(0) = -\alpha_i L + 2\sum_{j=1}^{n} G_{ij}\sqrt{c_j}\|\cosh u_j\|_1$$

where $\|\cosh u_j\|_1$ is the L^1 norm of $\cosh u_i$ in $[0, L]$ Hence we see that

$$\forall\, i = 1, \ldots, n \qquad 2\sum_{j=1}^{n} G_{ij}\sqrt{c_j}\|\cosh u_j\|_1 = -\tfrac{1}{2}\log R_i^L - \tfrac{1}{2}\log R_i^0 + \alpha_i L, \tag{5.4}$$

We define the vector $\mu \in \mathbb{R}^n$ by

$$\mu_i = -\tfrac{1}{2}\log R_i^L - \tfrac{1}{2}\log R_i^0 + \alpha_i L \quad \text{for} \quad i = 1, \ldots, n.$$

If G is invertible, we set $q = G^{-1}\mu \in \mathbb{R}^n$. The equation (5.4) then writes

$$\forall\, i = 1, \ldots, n \qquad \sqrt{c_i}\|\cosh u_i\|_1 = \tfrac{1}{2}q_i. \tag{5.5}$$

This shows that the condition $q_i > 0$ is necessary to have the existence of a solution satisfying $c_i > 0$. Moreover, under this condition, the system (5.1) is equivalent to the system

$$\forall\, i = 1, \ldots, n \qquad u_i' = -\alpha_i + \sum_{j=1}^{n} G_{ij} q_j \frac{\cosh u_j}{\|\cosh u_j\|_1}. \tag{5.6}$$

Note that this equation is not an ordinary differential equation, and that it can be set for arbitrary, possibly negative, q_i. The condition $q_i > 0$ for all $i = 1, \ldots, n$ thus appears as a conditions such that the problem (5.1-5.2-5.3) admits a "physical" solution with non negative c_i.

Note moreover that if $u = (u_i)_{i=1}^n$ satisfies (5.6) with the boundary conditions (5.2), then by definition of q, the boundary conditions (5.3) are also satisfied.

We begin by showing an existence result for a general problem of the form (5.6) with Cauchy boundary conditions in $x = 0$ (see [3]):

Proposition 5.1 *Let $\beta \in \mathbb{R}^n$, A be a matrix of size n, and $v^0 \in \mathbb{R}^n$. There exists a vector v with smooth coefficients $v_i(x)$, $i = 1, \ldots, n$ defined on $[0, L]$, solution of the equations:*

$$\textit{For} \quad i = 1, \ldots, n \qquad \begin{cases} v_i'(x) &= \beta_i + \displaystyle\sum_{j=1}^{n} A_{ij} \frac{\cosh v_j(x)}{\|\cosh v_j\|_1}, \quad \textit{for} \quad x \in [0, L], \\ v_i(0) &= v_i^0. \end{cases} \tag{5.7}$$

Using this result with $\beta_i = -\alpha_i$, $v_i^0 = \frac{1}{2}\log R_i^0$ for $i = 1, \ldots, n$ and $A_{ij} = G_{ij} q_j$ for $i, j = 1, \ldots, n$, we deduce immediatly the following result:

Theorem 5.2 *Suppose that n is even. Let μ the n-vector defined by*

$$\mu_i = -\tfrac{1}{2}\log R_i^L - \tfrac{1}{2}\log R_i^0 + \alpha_i L \quad \textit{for} \quad i = 1, \ldots, n. \tag{5.8}$$

The system (5.1) *together with the boundary conditions* (5.2) *and* (5.3) *possesses a smooth solution with $c_i > 0$ for $i = 1, \ldots, n$, if and only if the vector $q = G^{-1}\mu$ satisfies*

$$q_i > 0 \quad \textit{for} \quad i = 1, \ldots, n. \tag{5.9}$$

In this case, we have the relation

$$c_i = \left(\frac{q_i}{2\|\cosh u_i\|_1}\right)^2 \quad \textit{for} \quad i = 1, \ldots, n.$$

We will discuss later the condition (5.9) and show that it is satisfied in cases of practical interest (see Proposition 6.3 below).

PROOF OF PROPOSITION 5.1. We see that v is solution of (5.7) if and only if it is solution of the equation

$$v = \Phi(v)$$

where Φ is defined by

$$\Phi(v)_i(x) = v_i^0 + \beta_i x + \sum_{j=1}^n A_{ij} \int_0^x \frac{\cosh v_j(t)}{\|\cosh v_j\|_1} \mathrm{d}t, \quad \text{for} \quad i = 1, \dots, n. \tag{5.10}$$

On the space $(\mathrm{L}^\infty)^n$, we define the norm $\|v\|_\infty = \max_{i=1}^n \|v_i\|_{\mathrm{L}^\infty}$. If $v \in (\mathrm{L}^\infty)^n$, we see that have for all $x \in [0, L]$,

$$\|\Phi(v)\|_\infty \leq \|v^0\|_\infty + L\|\beta\|_\infty + \|A\|_\infty, \tag{5.11}$$

where $\|\cdot\|_\infty$ denotes either the norm on $(\mathrm{L}^\infty)^n$ or the standard infinity norm for vector and matrix in $\mathbb{R}^n$. Note that we used the fact that

$$0 \leq \int_0^x \frac{\cosh v_j}{\|\cosh v_j\|_1} \leq 1. \tag{5.12}$$

Moreover, using the fact that for all $j = 1, \dots, n$, $\|\cosh v_j\|_1 \geq L$, we also have for all $v \in (\mathrm{L}^\infty)^n$ and $i = 1, \dots, n$,

$$\left| \frac{\mathrm{d}\Phi(v)_i(x)}{\mathrm{d}x} \right| \leq |\beta_i| + \frac{1}{L} \sum_{j=1}^n |A_{ij}| \cosh \|v_j\|_{\mathrm{L}^\infty}$$

and hence

$$\|\Phi(v)'\|_\infty \leq \|\beta\|_\infty + \frac{1}{L}\|A\|_\infty \cosh \|v\|_\infty.$$

Thus Φ defines a function from $(\mathrm{L}^\infty)^n$ to $(\mathrm{W}^{1,\infty})^n$. Now we compute that for u and v in $\left(\mathrm{L}^\infty\right)^n$, we have for all $i = 1, \dots, n$,

$$\begin{aligned} \Phi(u)_i(x) - \Phi(v)_i(x) = &\sum_{j=1}^n A_{ij} \frac{1}{\|\cosh u_j\|_1} \int_0^x \big(\cosh u_j(t) - \cosh v_j(t)\big)\,\mathrm{d}t \\ &+ \sum_{j=1}^n A_{ij} \left(\int_0^x \frac{\cosh v_j(s)}{\|\cosh v_j\|_1} \mathrm{d}s \right) \frac{1}{\|\cosh u_j\|_1} \int_0^L \big(\cosh v_j(t) - \cosh u_j(t)\big)\,\mathrm{d}t. \end{aligned} \tag{5.13}$$

Using the fact that the L^1 norms of the $\cosh u_j$ and $\cosh v_j$ are greater than L, and (5.12) we get the bound:

$$|\Phi(u)_i(x) - \Phi(v)_i(x)| \leq 2 \sum_{j=1}^n |A_{ij}| \|\cosh u_j - \cosh v_j\|_{\mathrm{L}^\infty},$$

for all $i = 1, \dots, n$. Now we see that if u and v satisfy $\|u\|_\infty \leq M$ and $\|v\|_\infty \leq M$, we have

$$\|\Phi(u) - \Phi(v)\|_\infty \leq 2\|A\|_\infty (\sinh M)\|u - v\|_\infty. \tag{5.14}$$

In a similar way, we find that

$$\|\Phi(u)' - \Phi(v)'\|_\infty \leq \frac{1}{L}\left(1 + \frac{\cosh M}{L}\right) \|A\|_\infty (\sinh M)\|u - v\|_\infty.$$

This shows that Φ is continuous from $(\mathrm{L}^\infty)^n$ to $(\mathrm{W}^{1,\infty})^n$.

As $[0, L]$ is bounded, the injection $\mathrm{W}^{1,\infty} \to C(0, L)$ is compact. Thus Φ defines a continuous compact application

$$\Phi : \left(\mathrm{L}^\infty\right)^n \longrightarrow \left(\mathrm{L}^\infty\right)^n$$

with bounded image $K \subset C(0, L)^n$ (see (5.11)). By the Schauder theorem, Φ has a fixed point v in K.

Thus v is a continuous solution of (5.7) and we see by induction that $v \in C^\infty(0, L)$. This shows the proposition. ∎

Remark 5.3 In the case where $\beta_i = 0$, the system (5.7) is invariant by scaling of the interval $[0, L]$. Indeed, we see that if the functions v_i, $i = 1, \dots, n$, are solutions of (5.7) on $[0, L]$, then the functions $y \mapsto v_i(Ly)$ are solutions of the same equation on $(0, 1)$ (with the L^1 norm on $(0, 1)$. This is easily seen by change of variable. ∎

In [3] we prove that for $n = 2$, $\alpha_i = 0$ and G skew-symmetric, the solution given in Theorem 5.2 is unique. As we will see now, uniqueness for higher dimension is only proved here under smallness hypothesis on the data. As before, we first give a general proposition for problem of the form (5.7):

Proposition 5.4 *There exists an number $\varepsilon > 0$ such that for all matrix A and vectors v^0 and β satifying*

$$\|v^0\|_\infty + L\|\beta\|_\infty + \|A\|_\infty < \varepsilon, \tag{5.15}$$

the solution v of Proposition 5.1 is unique. Moreover, the sequence $v^{(k)}$ for $k \geq 0$ defined by $v^{(k+1)} = \Phi(v^{(k)})$ where Φ is defined by (5.10) and $v^{(0)} = 0$ converges toward v.

The multiplication by L in the inequality (5.15) is only due to homogeneity reasons (see (5.11)). Before proving this result, we show how it yields a uniqueness result for the problem (5.1-5.2-5.3). Recall that for given positive numbers R_i^0, R_i^L and α_i we set $\mu_i = -\frac{1}{2}\log R_i^L - \frac{1}{2}\log R_i^0 + \alpha_i L$ for $i = 1, \dots, n$ and $q = G^{-1}\mu$. For $\delta > 0$, we define the set

$$\Omega_\delta = \{ \quad R_i^0, R_i^L, \alpha_i \in \mathbb{R}^{3n} \quad | \quad |R_i^0 - 1| < \delta, \quad |R_i^L - 1| < \delta \quad \text{and} \quad |\alpha_i| < \delta \quad \} \tag{5.16}$$

Note that for all $\delta > 0$, Ω_δ is a domain of $\mathbb{R}^{3n}$ containing the point $(1, 1, 0)$ (i.e. $R_i^1 = 1$, $R_i^L = 1$ and $\alpha_i = 0$ for all $i = 1, \dots, n$). Using Proposition 5.4 with $\beta_i = -\alpha_i$, $v_i^0 = \frac{1}{2}\log R_i^0$ for $i = 1, \dots, n$ and $A_{ij} = G_{ij}q_j$ for $i, j = 1, \dots, n$, we deduce the following result:

Theorem 5.5 *There exist a real number δ such that if $(R_i^0, R_i^L, \alpha_i) \in \Omega_\delta$ satisfies*

$$\forall\, i = 1, \dots, n \quad q_i > 0$$

where

$$q = G^{-1}\mu \quad \textit{with} \quad \forall\, i = 1, \dots, n \quad \mu_i = -\tfrac{1}{2}\log R_i^L - \tfrac{1}{2}\log R_i^0 + \alpha_i L,$$

then the solution of the equation (5.1-5.2-5.3) is unique. Moreover, the function $u^{(k)}$ given by the following algorithm converges toward this solution: For $i = 1, \dots, n$, we define the sequence $u_i^{(k)}(x)$ of functions on $[0, L]$ by $u_i^{(0)}(x) = 0$ for $x \in [0, L]$, and for all $k > 1$, $u_i^{(k)}(x)$ satisfies the equations, for $i = 1, \dots, n$

$$\begin{aligned} \left(u_i^{(k+1)}\right)' &= -\alpha_i + \sum_{j=1}^{n} G_{ij} q_j \frac{\cosh u_j^{(k)}}{\|\cosh u_j^{(k)}\|_1}, \\ u_i^{(k+1)}(0) &= \tfrac{1}{2}\log R_i^0. \end{aligned} \tag{5.17}$$

PROOF OF PROPOSITION 5.4. Under the assumption (5.15) and using (5.11), we see that for all $v \in (L^\infty)^n$, $\Phi(v)$ takes values in the ball $B_\infty(0,\varepsilon)$ of $(L^\infty)^n$. If v and u are solutions of (5.7), they thus satisfy $\|v\|_\infty < \varepsilon$ and $\|u\|_\infty < \varepsilon$.

Using the equation (5.14), we thus have, as $\|A\|_\infty < \epsilon$,

$$\|\Phi(u) - \Phi(v)\|_\infty \leq 2\varepsilon(\sinh\varepsilon)\|u - v\|_\infty. \tag{5.18}$$

Let $k = 2\varepsilon(\sinh\varepsilon)$. It is clear that for ε sufficiently small, we have $k < 1$. This means that $u = v$.

Now we verify that for ε sufficiently small, Φ is contractant from $B_\infty(0,\varepsilon)$ to itself, and thus, that the sequence $v^{(k)}$ defined in the theorem converges toward the unique solution $v = \Phi(v)$. ■

Remark 5.6 In [3], in the case where $\beta = 0$, we show a slightly different result: For a given v^0, there exist and ε such that if $\|A\|_\infty < \varepsilon$, the solution is unique. Here, as we want to solve the original problem with a given pump power P, we need to have uniform estimate with respect to the initial value v^0. ■

6 EXISTENCE AND UNIQUENESS RESULTS FOR n EVEN: THE INITIAL EQUATIONS

Comparing the equations (4.2-4.3-4.4) and (5.1-5.2-5.3) we see that solving the first system is equivalent to find an R_{in} solution of the equations $\sqrt{R_{\text{in}}} = P/\sqrt{c_1}$ where c_1 depends on R_{in}, for fixed α_i, R_i^L $(i = 1, \dots, n)$ and R_i^0 $(i = 2, \dots, n)$ and P. Using the relation (5.5), this writes

$$R_{\text{in}} = \frac{4P^2}{q_1^2}\|\cosh u_1\|_1^2 =: g(R_{\text{in}}). \tag{6.1}$$

Note that in order to define the function g, the data have to be taken in the set Ω_δ with δ sufficiently small.

Here, we show that under conditions on P and the datas, we can find a unique solution of this equation if the datas are "small": this mean in particular that the solution $R_{\text{in}} = g(R_{\text{in}})$ is close to 1. Equivalently, this means that $\sqrt{c_1}$ is close to P or $\frac{1}{2}q_1\|\cosh u_1\|_1^{-1}$ is close to P.

For $\delta > 0$, we introduce the following set:

$$U_\delta := \{ \quad (R_j^0)_{j=2}^n, \quad (R_i^L, \alpha_i)_{i=1}^n \quad | \quad |R_j^0 - 1| < 1, \quad |R_i^0 - 1| < 1, \quad \text{and} \quad |\alpha_i| < \delta \}. \tag{6.2}$$

We set $X = (R_2^0, \dots, R_n^0, R_1^L, \dots, R_n^L, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{3n-1}$ an element of U_δ. Note that if $|R_1^0 - 1| < \delta$ then $(R_1^0, X) \in \Omega_\delta$.

We write $q_j(R, X)$ for the coefficients of $q = G^{-1}\mu$ where μ depends on (R, X) via the equation (5.8) for $R_1^0 = R_{\text{in}}$.

We first note the following: suppose that $X \in U_\delta$ is fixed, then as $G_{11} = 0$, for all R_1^0, q_1 depends only on X, and is written $q_1(X)$.

Now for fixed $X \in U_\delta$, P is closed to $\sqrt{c_1}$ means that P is close to $\frac{1}{2}q_1(X)\|\cosh u_1\|_1^{-1}$. But if δ is sufficiently small, the solution u is close to zero, and thus $\|\cosh u_1\|_1$ is approximatively equal to L. Hence the condition on P has to be on the form: P is close to $\frac{q_1(X)}{2L}$. As we will see in the next Theorem, we indeed have existence and uniqueness of the solution under such a condition, up to some technical points.

Theorem 6.1 *There exists a real number $\delta > 0$, such that if $X \in U_\delta$ satisfies:*

$$q_1(X) > 0 \quad \textit{and} \quad \forall\, R \in I_\delta := [1, 1+\delta], \quad q_j(R, X) \geq 0 \quad \textit{for} \quad j = 2, \ldots, n, \tag{6.3}$$

and if P is a real number such that

$$\frac{q_1(X)}{2L} \leq P \leq \frac{q_1(X)}{2L}(1 + \frac{\delta}{4}), \tag{6.4}$$

then there exists a unique $R_{\rm in} \in [1, 1+\delta]$ such that the solution (u, c) of the equations (5.1-5.2-5.3) *for the data $R_{\rm in}$ and X satisfies*

$$R_{\rm in} = \frac{P^2}{c_1} = \frac{4P^2}{q_1^2(X)} \| \cosh u_1 \|_1^2,$$

and thus (u, c) is a unique solution of the initial problem (4.2-4.3-4.4)*. If moreover we have $q_j(R_{\rm in}, X) > 0$ for $j = 2, \ldots, n$, then (F, B) defined by $F_i = \sqrt{c_i} e^{u_i}$ and $B_i = \sqrt{c_i} e^{-u_i}$ for $i = 1, \ldots, n$ is the unique positive solution of* (2.1-2.2-2.3)*.*

Proof. Let $X \in U_\delta$ and $R \in I_\delta := [1, 1+\delta]$. We recall that for δ sufficiently small, the function Φ defined in (5.10) satisfies the following estimates: There exists continuous functions $M(\delta) > 0$ and $\rho(\delta) > 0$ such that $M(\delta) \to 0$, $\rho(\delta) \to 0$ as $\delta \to 0$, and such that

$$\forall\, u, v \in (\mathrm{L}^\infty)^n \quad \Phi(u) \in B_\infty(0, M(\delta)) \quad \text{and} \quad \|\Phi(u) - \Phi(v)\|_\infty \leq \rho(\delta) \|u - v\|_\infty .$$

In the following, when X is fixed, we write $\Phi(u, R)$ instead of $\Phi(u)$ in order to precise the value of $R = R_1^0$.

Lemma 6.2 *There exists a $\delta_0 > 0$ such that for $\delta < \delta_0$ and for $X \in U_\delta$, $R \in I_\delta$, $\tilde{R} \in I_\delta$, if u and $\tilde{u}$ are the unique solutions of*

$$u = \Phi(u, R) \quad \textit{and} \quad \tilde{u} = \Phi(\tilde{u}, R),$$

then we have

$$\|u - \tilde{u}\|_\infty \leq C(G) |R - \tilde{R}\,|$$

where $C(G)$ depends only on G.

We postpone the proof of this lemma. Let $\delta < \delta_0$, and let $X \in U_\delta$ satisfying (6.3). For $R \in I_\delta$, we define the function

$$g(R) = \frac{4P^2}{q_1(X)^2} \| \cosh u_1 \|_1^2$$

where u is solution of $u = \Phi(u, R)$. Suppose that P satisfies (6.4). Then we have for $R \in I_\delta$,

$$g(R) \geq \frac{4L^2 P^2}{q_1(X)^2} \geq 1.$$

Moreover, as X satisfies (6.3), $\alpha_1 \geq 0$, and the coefficients $G_{1j} \leq 0$ for $j = 2, \ldots, n$, we have $u_1'(x) \leq 0$ for $x \in [0, L]$ (see (5.6)) and thus $u_1(x) \leq u_1(0) = \log \sqrt{R}$. Thus

$$g(R) \leq \frac{4L^2 P^2}{q_1(X)^2} (\cosh u_1(0))^2.$$

But we have

$$\cosh u_1(0) = \frac{R + 1}{2\sqrt{R}}.$$

Thus, we compute that if $R \in [1, 1+\delta]$ we have

$$g(R) \leq \frac{L^2P^2(2+\delta)^2}{q_1(X)^2(1+\delta)}.$$

Under the condition (6.4), we see that

$$g(R) \leq \frac{(2+\delta)^2}{4(1+\delta)}\left(1+\tfrac{\delta}{4}\right)^2$$

But we easily see that if δ_0 is sufficiently small, we have for $\delta_0 > \delta > 0$,

$$(2+\delta)^2\left(1+\tfrac{\delta}{4}\right)^2 \leq 4(1+\delta)^2$$

and thus

$$g(R) \leq 1+\delta$$

for $0 < \delta < \delta_0$ and $1 \leq R \leq 1+\delta$.

Hence, under the condition (6.4), g maps I_δ to itself.

Now we compute that for R and $\tilde{R}$ in I_δ,

$$g(R) - g(\tilde{R}) = \frac{4P^2}{q_1(X)^2}\left(\|\cosh u_1\|_1^2 - \|\cosh \tilde{u}_1\|_1^2\right)$$

where u and $\tilde{u}$ are the solutions of $u = \Phi(u, R)$ and $\tilde{u} = \Phi(\tilde{u}, \tilde{R})$ respectively. But we have $\|u\|_\infty \leq M(\delta)$ and $\|\tilde{u}\|_\infty \leq M(\delta)$, thus we have

$$|g(R) - g(\tilde{R})| \leq \frac{4L^2P^2}{q_1(X)^2} 2\left(\cosh M(\delta)\right)\left(\sinh M(\delta)\right)\|u-\tilde{u}\|_\infty.$$

But using the bound (5.9) and the lemma 6.2, we have

$$|g(R) - g(\tilde{R})| \leq 2C(G)(1+\tfrac{\delta}{4})^2(\cosh M(\delta))(\sinh M(\delta))|R-\tilde{R}|.$$

Thus, as $M(\delta)$ tends to 0 as $\delta \to 0$, we can suppose that δ_0 is sufficiently small such that for $\delta < \delta_0$, g is contractant from I_δ to itself.

Hence, there exists a unique fixed point for g, and this yields the result. ∎

Proof of lemma 6.2. Let δ_0 be such that $\rho(\delta) \leq \frac{1}{2}$ for $\delta \leq \delta_0$. As u and $\tilde{u}$ are in $B_\infty(0, M(\delta))$, we compute directly that

$$\begin{aligned}
\|u-\tilde{u}\|_\infty &= \|\Phi(u,R) - \Phi(\tilde{u},\tilde{R})\|_\infty \\
&\leq \|\Phi(u,R) - \Phi(\tilde{u},R)\|_\infty + \|\Phi(\tilde{u},R) - \Phi(\tilde{u},\tilde{R})\|_\infty \\
&\leq \tfrac{1}{2}\|u-\tilde{u}\|_\infty + \|\Phi(\tilde{u},R) - \Phi(\tilde{u},\tilde{R})\|_\infty.
\end{aligned}$$

Thus we have, using the expression of Φ,

$$\|u-\tilde{u}\|_\infty \leq |\log R - \log \tilde{R}| + 2\sum_{j=2}^{n} |q_j - \tilde{q}_j||G_{ij}|,$$

where q_j and $\tilde{q}_j$ denote the values of q depending on R and $\tilde{R}$ for fixed X (recall that $q_1(X)$ does not depend on R). As $q = G^{-1}\mu$ with μ defined in (5.8), we see that there exits $C(G)$ depending only on G such that

$$\|u-\tilde{u}\|_\infty \leq C(G)|\log R - \log \tilde{R}|.$$

But as R and $\tilde{R}$ are in I_δ, we have $|\log R - \log \tilde{R}| \leq |R - \tilde{R}|$. Thus we get the result. ■

We conclude this section by showing that the condition (5.9) and (6.3) are satisfied in an ideal situation where G has only non zero coefficient on the upper and lower first diagonals, the α_i are zero, and most of the reflectivity coefficients R_i^0 and R_i^L are equals to 1.

Proposition 6.3 *Suppose that G satisfies*

$$G_{ij} = 0 \quad \textit{for} \quad |i-j| < 1. \tag{6.5}$$

Suppose moreover that for all $i = 1, \ldots, n$, $\alpha_i = 0$,

$$\forall\, i = 2, \ldots, n, \quad R_i^0 = 0 \qquad \textit{and} \qquad \forall\, i = 1, \ldots, n-1, \quad R_i^L = 0,$$

then for all $R_{\text{out}} =: R_n^L < 1$ and $R_{\text{in}} > 1$, if μ is defined in (5.8) and $q = G^{-1}\mu$, then we have $q_j > 0$ for all $j = 1, \ldots, n$. In particular, the condition (5.9) is satisfied. If $R_{\text{out}} =: R_n^L < 1$, then the condition (6.3) is satisfied.

PROOF. Under the hypothesis of the proposition, we have $\mu_i = 0$ for $i = 2, \ldots, n-1$, $\mu_1 = -\frac{1}{2}\log R_{\text{in}}$ and $\mu_n = -\frac{1}{2}\log R_{\text{out}}$. Under the condition (6.5), the equation $q = G^{-1}\mu$ writes

$$q_i = \begin{cases} \nu_i(\frac{1}{2}\log R_{\text{in}}) & \text{for} \quad i \quad \text{even}, \\ \nu_i(-\frac{1}{2}\log R_{\text{out}}) & \text{for} \quad i \quad \text{odd}, \end{cases}$$

where ν_i are positive numbers. We deduce immediatly the result from these formulas. ■

7 Existence and uniqueness results for n odd

In this section, we will suppose that $n = 2p + 1$. In practical situations, the matrix G is of rank $2p$, and there exists always a vector a such that $a^T G = 0$ and a vector b such that $Gb = 0$ (if G is skew symmetric, we can take $a = b$).

We will prove in this section existence and uniqueness result for the system (4.2) together with the initial boundary conditions (4.3-4.4) under smallness assumptions on the data. Note that if we consider the boundary conditions (5.2-5.3), the system for n odd cannot be directly written as a system (5.6) because the matrix G is not invertible.

However, there exists by hypothesis a vector $a = (a_i)_{i=1}^n$ such that $a^T G = 0$ and a vector $b = (b_i)_{i=1}^n$ such that $Gb = 0$. In the following, we will suppose that the first coefficients a_1 and b_1 of a and b do not vanish.

When $\alpha_i = 0$ for all i, following the proof of Proposition 3.3, we compute that in the initial coordinate system (F, B), the function $\sum_{i=1}^n a_i \log F_i$ is an invariant of the system.

When $\alpha_i \neq 0$, it is clear that we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{i=1}^n a_i \log F_i\right) = -\sum_{i=1}^n a_i \alpha_i$$

and thus we get the relation

$$\sum_{i=1}^n a_i \left(\log F_i(L) - \log F_i(0)\right) = -\sum_{i=1}^n a_i \alpha_i L. \tag{7.1}$$

Recall that after the change of coordinates (4.1), we have

$$\log F_i(L) - \log F_i(0) = u_i(L) - u_i(0).$$

Inserting the boundary conditions (4.3) and (4.4) in (7.1) we see that

$$\sum_{i=1}^{n} a_i \left(u_i(L) - u_i(0) + \alpha_i L\right) = 0.$$

If we suppose that $a_1 \neq 0$, the previous relation implies that

$$u_1(0) = u_1(L) + \alpha_1 L + \sum_{i=2}^{n} \frac{a_i}{a_1} \left(u_i(L) - u_i(0) + \alpha_i L\right), \tag{7.2}$$

and thus, as $u_1(0) = \log P - \log \sqrt{c_1}$, we find

$$\log \sqrt{c_1} = \log P + \tfrac{1}{2} \log R_1^L - \alpha_1 L + \textstyle\sum_{i=2}^{n} \frac{a_i}{a_1} \left(\tfrac{1}{2} \log(R_i^L R_i^0) - \alpha_i L\right) \tag{7.3}$$

with the notation $R_n^L = R_{\text{out}}$.

This shows that in the odd case, the constant c_1 is determined analytically from the parameters of the problem. Thus, we can consider that the boundary condition in $u_1(0)$ is of the form $u_1(0) = \log \sqrt{R_1^0}$ where $R_1^0 = P^2/c_1$ is given.

Consider now the system (4.2) together with the boundary conditions (4.3) and (4.4). Integrating from 0 to L, we get for $i = 1, \dots, n$,

$$u_i(L) - u_i(0) + \alpha_i L = 2 \sum_{j=1}^{n} G_{ij} \sqrt{c_j} \| \cosh U_j \|_1 .$$

This system can be written $Gq = \mu$ where $\mu_i = u_i(L) - u_i(0) + \alpha_i L$ and $q_i = 2\sqrt{c_i} \| \cosh u_i \|_1$ for $i = 1, \dots, n$. The matrix G is not invertible, but we know that the kernel of G^T is of dimension 1, spanned by the vector a. Moreover, the kernel of G^T is the orthogonal of the image of G. But the constant c_1 is such that the vector μ and a are orthogonal (see (7.2)). This means that μ is in the image of G, and that q is determined up to an element of the kernel of G.

Let q^0 be any particular solution of the system $Gq = \mu$. The solution q can thus be written

$$q = q^0 + \lambda b$$

where $\lambda \in \mathbb{R}$. Now writing the first component of this equation yields

$$2\sqrt{c_1} \| \cosh u_1 \|_1 = q_1^0 + \lambda b_1,$$

and thus we get

$$\lambda = \frac{1}{b_1} \left(2\sqrt{c_1} \| \cosh u_1 \|_1 - q_1^0 \right) = \frac{1}{b_1} \left(2 \frac{P}{\sqrt{R_{\text{in}}}} \| \cosh u_1 \|_1 - q_1^0 \right).$$

where $R_{\text{in}} = R_1^0$ is fixed. We summarize these result as follows:

Proposition 7.1 *Suppose that n is odd, G is of rank $2p$, there exists a vector $a = (a_i)_{i=1}^n$ such that $a_1 = 1$ and $a^T G = 0$, and there exists a vector $b = (b_i)_{i=1}^n$ such that $b_1 = 1$ and $Gb = 0$. Then the system (4.2-4.3-4.4) is equivalent to the system in u and λ:*

$$\begin{array}{rcll} u_i'(x) & = & -\alpha_i + \displaystyle\sum_{j=1}^{n} G_{ij} (q_j^0 + \lambda b_j) \frac{\cosh u_j(x)}{\| \cosh u_j \|_1} & \textit{for} \quad i = 1, \dots, n \quad \textit{and} \quad x \in [0, L] \\ \lambda & = & \frac{2P}{\sqrt{R_1^0}} \| \cosh u_1 \|_1 - q_1^0 & \\ u_i(0) & = & \frac{1}{2} \log R_i^0 & \textit{for} \quad i = 1, \dots, n. \end{array} \tag{7.4}$$

where

$$R_1^0 = e^{(2L\sum_{i=1}^n a_i\alpha_i)}(R_1^L)^{-1}\prod_{i=2}^n (R_i^L R_i^0)^{-a_i} \tag{7.5}$$

depends only on the data of the problem, and where q^0 is a given particular solution of $Gq = \mu$ with μ defined as usual by

$$\mu_i = -\tfrac{1}{2}\log R_i^L - \tfrac{1}{2}\log R_i^0 + \alpha_i L.$$

Recall that X denotes an element of U_δ defined in (6.2). As R_1^0 is given by (7.5), we see that the vector q_i^0 depend only on X, and we write $q_i^0(X)$. Note that if $X \in U_\delta$ with δ sufficiently small, the number R_1^0 is close to 1 and the vector q^0 is closed to 0. Using this, we can prove the following result:

Theorem 7.2 *There exists a real number δ_0 such that for all $\delta < \delta_0$, if $X \in U_\delta$ is such that*

$$\forall\, \lambda \in [0,\delta], \quad q_i^0(X) + \lambda b_i > 0 \quad \textit{for} \quad i = 1,\ldots,n,$$

then if P is a real number such that

$$\frac{q_1^0(X)\sqrt{R_1^0}}{2L} \le P \le (q_1^0(X) + \delta)\frac{R_1^0}{L(R_1^0+1)}, \tag{7.6}$$

where R_1^0 is defined by (7.5), then the problem (7.4) has a unique solution (u,λ) with $\lambda \in [0,\delta]$. Moreover, u is the solution of the system (4.2-4.3-4.4) with

$$\sqrt{c_i} = \frac{q_i^0 + \lambda b_i}{2\|\cosh u_i\|_1} \quad \textit{for} \quad i = 2,\ldots,n$$

and c_1 given by (7.3).

Proof. For any $\lambda \in \mathbb{R}$ and $X \in U_\delta$, we define the application Φ_λ as

$$\Phi_\lambda(v)_i(x) = \tfrac{1}{2}\log R_i^0 - \alpha_i x + \sum_{j=1}^n G_{ij}(q_i^0(X) + \lambda b_i)\int_0^x \frac{\cosh v_j(t)}{\|\cosh v_j\|_1}\,\mathrm{d}t, \quad \text{for} \quad i = 1,\ldots,n. \tag{7.7}$$

We state the following result, which is a consequence of the bounds (5.11) and (5.14).

Lemma 7.3 *There exist $\delta_0 > 0$ and there exist continuous functions $M(\delta) > 0$ and $\rho(\delta) > 0$ satisfying $M(\delta) \to 0$ and $\rho(\delta) \to 0$ as $\delta \to 0$, such that for all $\delta < \delta_0$, if $X \in U_\delta$ then for all $\lambda \in [0,\delta]$, Φ_λ satisfies*

$$\|\Phi_\lambda(u) - \Phi_\lambda(v)\|_\infty \le \rho(\delta)\|u - v\|_\infty$$

for u and v in $B_\infty(0, M(\delta)) \subset (\mathrm{L}^\infty)^n$

Now, if $\delta < \delta_0$ and $X \in U_\delta$, then for all $\lambda \in [0,\delta]$, there exists a unique $u^{(\lambda)}$ solution of $u^{(\lambda)} = \Phi_\lambda(u^{(\lambda)})$. Thus, we can define the function

$$g(\lambda) = \frac{2}{\sqrt{R_1^0}}P\|\cosh u_1^{(\lambda)}\|_1 - q_1^0(X). \tag{7.8}$$

The problem is reduced to show that g has a unique fixed point in $[0,\delta]$. But the condition (7.6) implies by similar computations as in the proof of Theorem 6.1, that g maps $[0,\delta]$ into itself. Moreover, we see that for λ and $\tilde\lambda$ we have

$$|g(\lambda) - g(\tilde\lambda)| \le \frac{2}{\sqrt{R_1^0}}P(\sinh M(\delta))\|u^{(\lambda)} - u^{(\tilde\lambda)}\|_\infty.$$

By an argument similar to the one in lemma 6.2 we conclude that for δ_0 sufficiently small, the function g is contractant from $[0, \delta]$ to itself. This shows the result. ■

8 Numerical experiments

In this part, we solve the system (2.1-2.2-2.3) in a situation of practical interest. We will consider the case where $n = 4$ or $n = 5$. The length of the fiber L is taken equal to 100 meters. The reflectivity coefficients R_i^0 and R_i^L are taken equal to 0.99 except for the last one, $R_{\text{out}} = 0.1$. The matrix G of Raman gains is taken (up to terms of order 10^{-9}) as

$$G = 10^{-3} \begin{bmatrix} 0 & -5.354693 & -0.833641 & -0.165746 & -0.001215 \\ 5.109551 & 0 & -5.091333 & -0.800871 & -0.246770 \\ 0.757437 & 4.847864 & 0 & -4.883841 & -0.694188 \\ 0.143011 & 0.724173 & 4.637914 & 0 & -3.546259 \\ 0.001000 & 0.212878 & 0.628922 & 3.383213 & 0 \end{bmatrix}.$$

for $n = 5$ and the sub-matrix of order 4 made by raising the last column and line of this matrix when $n = 4$.

Similarly, the matrix of attenuation coefficients is taken as the sub-matrix of appropriate size of

$$\operatorname{diag}(\alpha) = 10^{-3} \begin{bmatrix} 0.388799 & 0 & 0 & 0 & 0 \\ 0 & 0.346712 & 0 & 0 & 0 \\ 0 & 0 & 0.296873 & 0 & 0 \\ 0 & 0 & 0 & 0.252234 & 0 \\ 0 & 0 & 0 & 0 & 0.218211 \end{bmatrix}.$$

Note (see [7],[1],[2]) that the matrix G is of the following form: there exists a lower triangular matrix L with zero coefficients on the diagonal, such that if D is the diagonal matrix with the frequencies $\nu_i > 0$ corresponding to the F_i and B_i on the diagonal, then G is of the form

$$G = L - DL^T D^{-1}.$$

This means that $G_{ij} = L_{ij}$ for $i > j$ and $G_{ij} = -L_{ji}\frac{\nu_i}{\nu_j}$ for $i < j$. We see in particular that GD is skew-symmetric, and thus G has always a non trivial kernel if n is odd.

We first consider the problem (5.1-5.2-5.3) for $n = 4$ with a given $R_{\text{in}} = 20$. We use the algorithm (5.17). We first compute directly that for $i = 1, \dots, 4$, we have $q_i > 0$. We use the Euler method with integration step $h = L/1000$, and the corresponding formula to compute the integral in (5.17). The solution is plotted in the left graphic in figure 1. In the right graphic in figure 1, we plotted the relative error in L^1 norm between the components of the solution u of (5.1-5.2-5.3) for $R_{\text{in}} = 20$ and the successive approximations $u^{(k)}$ given by (5.17). We see that the convergence is linear in this practical case. Numerically, we observe the same linear convergence for large R_{in}.

We consider now the numerical approximation of the problem (4.2-4.3-4.4) when $n = 4$ and $P = 5$. We approximate a fixed point of the function g defined in (6.1) by defining the sequence $R^{k+1} = g(R^k)$ and $R^0 = 20$. To compute the function g, we use the previous algorithm for the modified problem. The solution in the coordinate system (F, B) is plotted in the left graphic of figure 2. The fixed point is $R_{\text{in}} \simeq 43.708504$. Finally, we consider the case where $n = 5$. We compute that in the proposition 7.1 we can take

$$a \simeq \begin{bmatrix} 1 \\ -0.152781 \\ 1.066044 \\ -0.198392 \\ 1.551714 \end{bmatrix}, \quad b \simeq \begin{bmatrix} 1 \\ -0.145786 \\ 0.968595 \\ -0.171179 \\ 1.277317 \end{bmatrix}, \quad \text{and} \quad q^0 \simeq 10^2 \begin{bmatrix} 0.149596 \\ 3.435993 \\ -0.465061 \\ 3.352517 \\ 0 \end{bmatrix}.$$

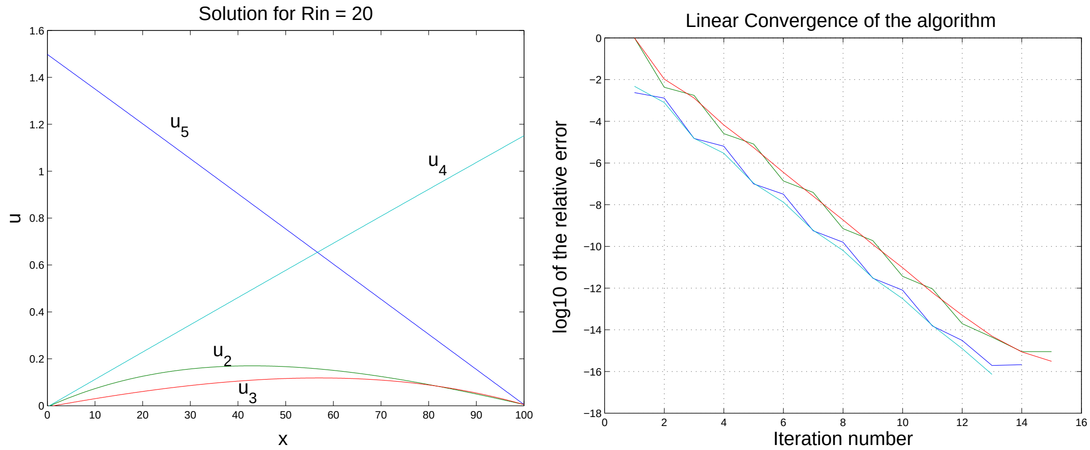

Figure 1

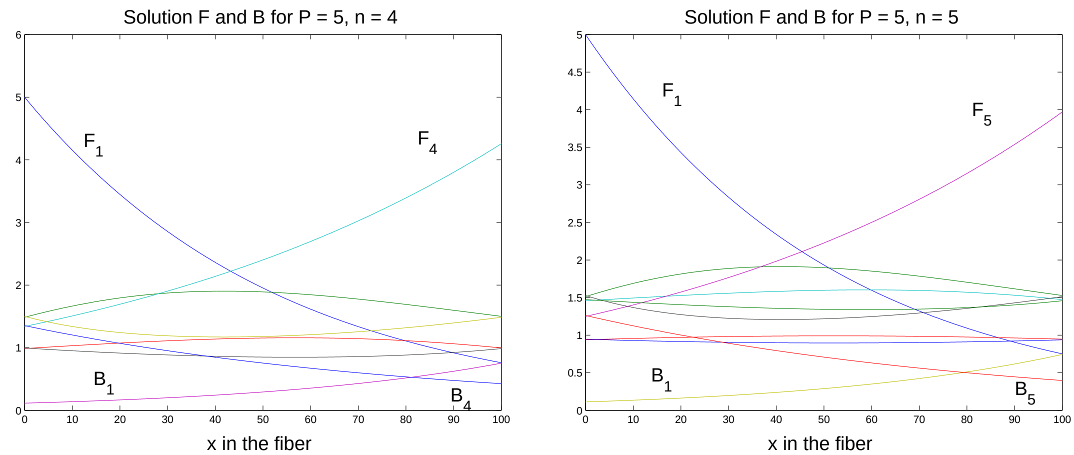

Figure 2

In the right graphic of figure 2 we plot the solution (F, B) for $P = 5$. This numerical approximation is computed by defining a sequence $\lambda^{(k+1)} = g(\lambda^{(k)})$ where this time g is given by the function (7.8). The solution is $\lambda \simeq 242.0841567$.

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Contents

Unité de recherche INRIA Rennes
IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)

Unité de recherche INRIA Futurs : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
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